

GEOMETRIC TATE-SWAN COHOMOLOGY OF EQUIVARIANT SPECTRA

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ABSTRACT. We sketch a quick and dirty geometric approach to the Tate-Swan cohomology of equivariant spectra, illustrating it with conjectural applications to Atiyah-Segal K -theory of circle actions, and a possible geometric model for the topological cyclic homology of the sphere spectrum.

For Michael and Graeme

1. INTRODUCTION

These variations on the theme of Tate-Swan cohomology of spectra are very old-fashioned, but my hope is to illustrate the flexibility and naturality of geometric methods. The first section below summarizes the general framework, while the second applies those ideas to Atiyah-Segal equivariant cohomology of circle actions. The third section proposes a model for the topological cyclic homology of the sphere spectrum.

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1.1 Let G be a (locally connected) d -dimensional compact Lie group, and let E_G be multiplicative cohomology theory, which admits a reasonable interpretation as a G -equivariant cobordism theory of manifolds with ‘ E -structure’ of some sort: the underlying non-equivariant theory could be, for example, the sphere spectrum, the noncommutative spectrum $M\xi$ of [2], MU , MO , or (depending on how hard we’re willing to work [1]) $H\mathbb{Z}$. It seems likely that some version of the work of Baum, Douglas [3] and others can also be handled by these methods.

We consider compact E -oriented manifolds Z with compatible G -structure, with boundary

$$\partial Z = \partial_0 Z \cup \partial_{\text{free}} Z$$

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partitioned into transversally intersecting parts, such that the G -action on $\partial_{\text{free}}Z$ is in fact free. I'll call such a manifold **closed** if $\partial_0Z = \emptyset$: in other words if the G -action on Z is free on the entire boundary. Two such manifolds will be said to be cobordant if there is a manifold W in this class, together with a transversal partition

$$\partial W = (Z_+ \sqcup Z_-^{\text{op}}) \cup V$$

with G -action free on V , and

$$V \cap (Z_+ \sqcup Z_-^{\text{op}}) = \partial Z_+ \sqcup \partial Z_-^{\text{op}} ;$$

this is all completely classical [6 I §4]. I'll write (some variant of) $t_n^G E$ for the abelian group of cobordism classes of such manifolds with group action free on the boundary, or more generally $t_*^G E(X)$ for the graded group of equivalence classes of manifolds mapped equivariantly to a pointed G -space X ; this defines a G -equivariant geometric homology theory. Stabilizing (ie taking the direct limit over suspensions from a suitable universe of G -representations [5]; in alternate terms, sheafifying a presheaf) defines, if we're lucky, a theory represented by a suitable G -spectrum¹.

1.2 For reasonable E -structures, the Cartesian product $Z \times Z'$ of two closed G -manifolds, given the diagonal G -action, can be smoothed to be another; more precisely, the boundary

$$\partial(Z \times Z') = \partial Z \times Z' \cup_{\partial Z \times \partial Z'} Z \times \partial Z'$$

has a natural smooth structure [6 I §3] compatible with the G -action. This defines an external product

$$t_n^G E(X) \otimes t_m^G E(Y) \rightarrow t_{n+m}^G E(X \wedge Y)$$

and, in particular, makes $t_*^G E(S^0)$ into a graded ring.

1.3 Examples

1) The interval $[-1, +1]$, regarded as an unoriented manifold with \mathbb{Z}_2 -action defined by $x \mapsto -x$, defines a class [4, 16-18]

$$w^{-1} \in t_1^{\mathbb{Z}_2} MO(S^0) .$$

2) The closed unit disk $D \subset \mathbb{C}$, regarded as a complex-oriented manifold with action

$$u, z \mapsto uz : \mathbb{T} \times D \rightarrow D$$

of the unit circle $\mathbb{T} = \partial D$, defines a class $c^{-1} \in t_2^{\mathbb{T}} MU(S^0)$.

3) The closed unit ball $B(\mathbb{H})$ in the quaternions, regarded as an \mathbb{H} -oriented manifold with action

$$SU(2) \times B(\mathbb{H}) \rightarrow B(\mathbb{H})$$

¹In particular, if we're not plagued by phantoms. I'm omitting details because the general constructions of [12] are quite convenient for these technical issues.

by multiplication of the unit quaternions, defines a class $\wp^{-1} \in t_4^{\mathrm{SU}(2)} M\mathrm{Sp}(S^0)$.

1.4 Constructions of Quillen (again, in good situations) associate to an n -manifold Z with **empty** boundary, an E -structure and a G -action, the class

$$[Z \times_G EG \rightarrow S^0 \wedge_G EG_+] \in E_G^{-n}(S^0)$$

of its homotopy quotient. Similarly, if the action of G on Z is in fact free, its geometric quotient

$$[Z/G \rightarrow BG_+] \in E_{n-d}(BG_+)$$

identifies Z as a principal G -bundle. Together these constructions fit into the fundamental exact sequence

$$\dots \xrightarrow{\phi} E_n^G(S^0) \xrightarrow{\rho} t_n^G E(S^0) \xrightarrow{\partial} E_{n-d-1}(BG_+) \longrightarrow \dots$$

of E_G^* -modules, with ρ a ring homomorphisms, and ϕ the forgetful map from free to unrestricted group actions. [The product of a general G -manifold and a manifold with free G -action, given the diagonal group action, is a manifold with free action.]

1.5.1 Examples, continued:

$$\begin{aligned} t_{\mathbb{Z}_2}^* H\mathbb{Z}_2(S^0) &= \mathbb{Z}_2[w^{\pm 1}] \\ t_{\mathbb{T}}^* H\mathbb{Z}(S^0) &= \mathbb{Z}[c^{\pm 1}] \\ t_{\mathrm{SU}(2)}^* H\mathbb{Z}(S^0) &= \mathbb{Z}[\wp^{\pm 1}] \end{aligned}$$

(where w^{-1} , c^{-1} and \wp^{-1} are the images of the corresponding cobordism classes under the Steenrod cycle map).

In particular, in

$$\dots \rightarrow H^{-*}(B\mathbb{T}_+, \mathbb{Z}) = \mathbb{Z}[c] \rightarrow t_{\mathbb{T}}^{-*} H\mathbb{Z}(S^0) = \mathbb{Z}[c^{\pm 1}] \rightarrow H_{*-2}(B\mathbb{T}_+, \mathbb{Z}) = \mathbb{Z}[\gamma_n | n \geq 0] \rightarrow \dots,$$

the product c^{-n} represents the class of the unit ball in \mathbb{C}^n , with \mathbb{T} acting as multiplication. The boundary map sends it to the divided power

$$\partial c^{-n} = [S^{2n-1}/\mathbb{T} = \mathbb{C}P^{n-1} \subset \mathbb{C}P^\infty = B\mathbb{T}] = \gamma_{n-1},$$

Kronecker dual to the $(n-1)$ st power of the usual first Chern class $c \in H^2(B\mathbb{T}, \mathbb{Z})$. The classes w^{-1} and \wp^{-1} are similarly related to the usual first Stiefel-Whitney and Pontrjagin classes.

1.5.2 More generally, $t_{\mathbb{T}}^* MU(S^0)$ is a formal Laurent series ring $MU^*((c))$, and $MU_* B\mathbb{T}$ is a free MU_* -module on generators β_n Kronecker dual to the Chern classes $c^n \in MU^{2n}(B\mathbb{T})$, satisfying

$$\beta(s_0 +_F s_1) = \beta(s_0)\beta(s_1),$$

where $\beta(s) = \sum \beta_n s^n$ and $s_0 +_F s_1 = F_{MU}(s_0, s_1)$ is the universal formal group law [19]; the argument above generalizes, implying $\partial c^{-n} = \beta_{n-1}$.

The map $\epsilon : B\mathbb{T} \rightarrow \text{pt}$ defines a kind of residue homomorphism

$$\text{res} : t_{\mathbb{T}}^* MU(S^0) \xrightarrow{\partial} MU_{-* -2} B\mathbb{T}_+ \xrightarrow{\epsilon_*} MU_{-* -2}(S^0)$$

satisfying $\text{res}(c^{-n}) = \delta_{n,1}$; this lets us write

$$(c^n, \beta_m) = \text{res}(c^n \cdot c^{-m-1}) \quad (n, m \geq 0)$$

for the Kronecker product, ie

$$\beta_m = \text{res}(c^{-m-1} \cdot -) \in MU_{2m} B\mathbb{T}_+ \rightarrow \text{Hom}_{MU}^{-2m}(MU^*(B\mathbb{T}_+), MU(S^0)) .$$

Thus if $f \in t_{\mathbb{T}}^* MU(S^0)$, $g \in MU^*(B\mathbb{T}_+)$, we have [17]

$$(\partial f)(g) = \text{res}(f \cdot g) \in MU_*(S^0) .$$

1.5.3 The Segal conjecture for finite groups [15] supplies another class of examples: after a suitable completion, the exact sequence above simplifies to an equivalence

$$t_{\hat{G}} \sim \bigvee_{e \neq H < G} \widehat{BW}_H$$

where $W_H = N(H)/H$ is a kind of Weyl group.

2. CLASSICAL $K_{\mathbb{T}}$

Rudimentary knowledge of equivariant homotopy theory tells us that an ordinary (nonequivariant) cohomology theory can have more than one equivariant extension, and the account above ignores this. Rather than confront that issue, I'll consider two examples related to this question, which I still do not understand well enough.

2.1 If G is a finite group, G. Wilson's identification [25 Prop 1.2] of $\tilde{K}_*(BG)$ as the \mathbb{Q}/\mathbb{Z} -dual of the augmentation ideal of the completed representation ring $\hat{R}(G)$, concentrated in odd degree, identifies [11, 12] the Tate cohomology

$$t_G K^{\text{hot}}(S^0) = \mathbb{Z} \oplus \hat{R}_+(G) \otimes \mathbb{Q}$$

of homotopy-theoretic G -equivariant K -theory (cf also [3]). It seems plausible that something similar holds for Atiyah-Segal equivariant K -theory, but I do not know of a proof. In any case Tate cohomology defines an interesting analog

$$K_G^{\text{hot}} \rightarrow t_G K^{\text{hot}}$$

of the Chern character.

2.2 The representation ring $R(\mathbb{T})$ of the circle is the Atiyah-Segal \mathbb{T} -equivariant K -theory $K[\chi^{\pm 1}]$ of a point ($K = \mathbb{Z}$ as a \mathbb{Z}_2 -graded ring, and $\chi = \exp(2\pi i\theta)$) is the standard character $\mathbb{T} = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$), while

$$K_{\mathbb{T}}^{\text{hot}}(S^0) := K(B\mathbb{T}_+) \cong \mathbb{Z}[[t]]$$

with $\chi = 1 - t$. The homology

$$K_*(B\mathbb{T}_+) = K[b_n \mid n \geq 0]$$

is the algebra (under Pontrjagin product) generated by elements satisfying the identity

$$b(s_0)b(s_1) = b(s_0 + s_1 - s_0s_1) ,$$

where

$$b(s) = (1 - s)^t = \sum_{n \geq 0} b_n s^n = \sum \binom{t}{n} (-s)^n$$

so the exact sequence

$$0 \rightarrow K(B\mathbb{T}_+) = K[[t]] \rightarrow t_{\mathbb{T}}K^{\text{hot}}(S^0) = \mathbb{Z}((t)) \rightarrow \mathbb{Z}[b_n \mid n \geq 0] \rightarrow 0$$

is much like Ex. 1.5.2.

2.3 The homomorphism $\chi \mapsto 1 - t$ defines a completion

$$t_{\mathbb{T}}K(S^0) \cong R(\mathbb{T})[(1 - \chi)^{-1}] \cong K[\chi^{\pm 1}, (1 - \chi)^{-1}] \rightarrow t_{\mathbb{T}}K^{\text{hot}}(S^0) = K((t))$$

at the identity $\chi = 1$ of the multiplicative group $\text{Spec } K_{\mathbb{T}} = \mathbb{G}_m$ (cf [22]), analogous to a point

$$\text{Spec } K((t)) \rightarrow \text{Spec } t_{\mathbb{T}}K(S^0) \sim \mathbb{P}^1 - \{0, 1, \infty\} .$$

The group

$$\Sigma_3 \cong \{\sigma, \tau \mid \tau^2 = 1, \sigma^3 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\}$$

acts by fractional linear transformations $\tau(\chi) = \chi^{-1}$, $\sigma(\chi) = (1 - \chi)^{-1}$ on the projective line over K , permuting the points $\{0, 1, \infty\}$. I don't know if this lifts to any kind of action on the functor $t_{\mathbb{T}}K$.

Writing $c = 2\pi i\theta$ extends the usual Chern character

$$K(B\mathbb{T}_+) \rightarrow \mathbb{Q}[[c]]$$

to a specialization $\text{Spec } \mathbb{Q}((c)) \rightarrow \text{Spec } t_{\mathbb{T}}K(S^0)$ of the map above, which sends $q := (1 - \chi^{-1})^{-1}$ to $c^{-1}(1 + \dots) \in c^{-1}\mathbb{Q}[[c]]$; while $q \mapsto \exp(\hbar)$ similarly maps χ to $\hbar^{-1}\mathbb{Q}[[\hbar]]$, defining another formal point

$$\text{Spec } \mathbb{Q}((\hbar)) \rightarrow \text{Spec } t_{\mathbb{T}}K(S^0) .$$

The first of these corresponds to completion near $c = 0$ (ie $\chi = 1$), while the second is completion near $\chi = \infty$. In terms of the distribution

$$\text{li}_1(x) = \log |1 - e^x| \ (\equiv -\log |x| \text{ mod smooth functions of } x)$$

defined by the composition

$$\log \circ \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \circ \exp$$

[9 §3 Cor 2] we have

$$-\hbar = \text{li}_1(-c), \quad -c = \text{li}_1(-\hbar) ,$$

suggesting that $\hbar \rightarrow 0 \iff c \rightarrow \infty$ is a kind of semiclassical limit ...

Note finally that $\mathbb{Z}[\chi^{\pm 1}, (1 - \chi)^{-1}]$ has Tate's Laurent series ring $\mathbb{Z}((\chi))$ (which accomodates K -theory with the σ -orientation) as a completion. This suggests the interest of the corresponding completion of $K_{\mathbb{T}}(LX)$ (or of $t_{\mathbb{T}}K(LX)$) as a model for elliptic cohomology [13]. On the other hand, a theorem of Goodwillie [10] suggests that $t_{\mathbb{T}}K^{\text{hot}}(LK)$ (and hence $K_{\mathbb{T}}(LX) \otimes_{K_{\mathbb{T}}} K\mathbb{Q}((c))$) sees only the fundamental group of X : the left vertical arrow in the diagram

$$\begin{array}{ccc} K_{\mathbb{T}}^{\text{hot}}(-)[c^{-1}] & \xrightarrow{ch} & H_{\mathbb{T}}^*(-, K\mathbb{Q})[c^{-1}] \\ \downarrow & \searrow & \uparrow \text{dotted} \\ K_{\mathbb{T}}(-)((c)) & \longrightarrow & K_{\mathbb{T}}^{\text{hot}}(-)[c^{-1}] \otimes \mathbb{Q} \end{array}$$

is an equivalence of functors, while the right vertical arrow is induced by the injective transformation

$$M \rightarrow M \otimes_{\mathbb{Z}((c)) \otimes \mathbb{Q}} \mathbb{Q}((c)) .$$

3. A MODEL FOR $\text{TC}(S^0)$

Work of Bökstedt, Hsiang, and Madsen identifies the topological cyclic homology $\text{TC}(S^0)$ of the sphere, after p -adic completion, with a similar completion of the spectrum $S^0 \vee \Sigma CP_{-1}^{\infty}$. This section records the construction of a geometric model TC^{geo} for the latter object, which has an interesting multiplicative structure. However we make no attempt to construct a map to or from $\text{TC}(S^0)$ itself.

3.1 A reasonable cobordism theory Ω_*^G has an associated cobordism theory $\Omega_*^G \oplus \Omega_{*-1}^G$ of G -manifolds with boundary [6 I §4], represented by $(S^0 \vee S^1) \wedge MG$; where MG is the spectrum representing Ω_*^G . The homotopy fiber TC^{geo} of the composition

$$\Sigma B\mathbb{T}_+ \rightarrow S^0 \rightarrow S^0 \vee S^1 = \mathbb{T}_+$$

(defined by the stable circle-transfer [14], followed by the obvious inclusion) can be interpreted as representing a cobordism theory of framed manifolds M with boundary, together with extra data defined by a complex line bundle on the manifold, and a trivialization of that bundle away from a collar neighborhood of a codimension zero submanifold of its boundary: a variation on the Baas-Sullivan theory [1] of cobordism with singularities, based on framed manifolds in which the boundary $\partial M = \partial_0 M \cup \partial_1 M$ is partitioned into two (transversally intersecting) parts, with $\partial_1 M$ carrying a line bundle trivialized away from $\partial(\partial_1 M)$. The operation $M \mapsto \partial_1 M$ thus satisfies $\partial_1 \circ \partial_1 = \emptyset$.

If M is closed in this relative sense (ie $\partial_1 M = \emptyset$), and

$$u, z \mapsto uz : \mathbb{T} \times D \rightarrow D$$

is the usual circle action on the disk $\{z \in \mathbb{C} \mid |z| = 1\}$, then (using the natural framing of the circle bundle $C(\partial_0 M)$)

$$D \times_{\mathbb{T}} C(\partial_0 M) \cup_{0 \times \partial_0 M} M := M_D$$

is a closed framed manifold. The closed objects in this relative cobordism category are thus something like the algebraic geometers' varieties bounded by divisors.

3.2 The homotopy exact sequence

$$\cdots \rightarrow \pi_{n+1}^S(S^0) \oplus \pi_n^S(S^0) \rightarrow \mathrm{TC}_n^{\mathrm{geo}}(S^0) \rightarrow \pi_{n-1}^S B\mathbb{T}_+ \rightarrow \pi_n^S(S^0) \oplus \pi_{n-1}^S(S^0) \rightarrow \cdots$$

associated to this construction starts at the left by sending a framed $(n+1)$ - manifold with boundary to its boundary, regarded as a framed manifold decorated with a trivial complex line bundle. The second arrow in the sequence sends a framed n - manifold with boundary and a suitable complex line bundle, to the class of its boundary, regarded as an element of the $(n-1)$ - dimensional bordism group of the classifying space for circle bundles. The circle transfer defines the third homomorphism of the sequence, which sends a closed framed $(n-1)$ - manifold with a circle bundle over it, to the total space of that bundle (given its natural framing), regarded as an n - manifold with empty boundary.

3.3 This cobordism theory has a natural multiplication, defined by the tensor product of line bundles over the cartesian product of underlying manifolds. $\mathrm{TC}_0^{\mathrm{geo}}$ is generated by a point, and $\mathrm{TC}_1^{\mathrm{geo}}$ has a rank one part generated by a closed interval. $\mathrm{TC}_{-1}^{\mathrm{geo}}$ has a somewhat unconvincing interpretation as generated by the -1 - dimensional manifold bounded by the -2 - dimensional manifold carrying the complex line bundle whose total space is a point \dots

3.4 It follows from the cofibration sequence

$$\cdots \rightarrow K^*(S^0 \vee S^1) \rightarrow K^*(\Sigma B\mathbb{T}_+) \rightarrow K^*(\mathrm{TC}^{\mathrm{geo}}) \rightarrow \cdots$$

that $K^0(\mathrm{TC}^{\mathrm{geo}}) \cong \mathbb{Z}$ and

$$K^1(\mathrm{TC}^{\mathrm{geo}}) \cong \mathbb{Z}\langle t^k \mid k \geq -1 \rangle$$

with $t = \eta - 1 : B\mathbb{T} \rightarrow BU$ classifying the Hopf line bundle. In the 1970's Segal suggested the homotopy fiber $\Sigma^{-1}K\mathbb{C}^\times$ of the map

$$K \xrightarrow{2\pi i} K\mathbb{C}$$

as an interesting model for the algebraic K -theory of \mathbb{C} (as a discrete field). This suggests regarding the image of t in $[\mathrm{TC}^{\mathrm{geo}}, \Sigma^{-1}K\mathbb{C}^\times]$ as a topological analog of Borel's regulator [8].

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